Housing Options: For Sale by Owner

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Abstract

In this paper I describe a methodology for pricing a set of contingent claims on the value of residential real estate. At least two such contracts are currently offered on owner-occupied single family homes and others have been proposed. I describe how to price these particular contracts, which are essentially call options written by the homeowner on the house value that come into existence when the home is sold. I use this pricing methodology to evaluate the fairness of these agreements for the homeowners writing the contracts. Furthermore, I discuss applications of these contracts related both to idiosyncratic housing price mitigation and well as restructuring mortgages affected by the current housing crisis.

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1 Introduction

The recent precipitous drop in U.S. housing values has raised two separate yet interconnected issues, each with its own affected clientele. For responsible homeowners, the key issue is how to mitigate the effect of housing price movements on their personal wealth. Another closely related issue is the growing number of mortgage defaults brought about to a large degree by the fall in housing prices. The key issue here is in finding a way to restructure these mortgages that is mutually beneficial to the lenders and the delinquent borrowers, and which properly aligns the incentives of these groups. In this paper, I examine a new type of contingent claim on residential real estate (the "Housing Option") which addresses both of these issues.

The first use for the Housing Option is as a way for a homeowner to hedge his exposure to real estate prices, a very timely topic in the current market. The topic of house price exposure is an important one for personal financial planning, and there is a substantial literature covering various aspects of the housing decision, including pricing of the options that a homeowner may hold, although these are primarily related to mortgages. Since the introduction of futures contracts on residential real estate indices there have been a number of papers on the potential benefit to homeowners of the use of such contracts. By contrast with the futures market, the Housing Option allows the homeowner to hedge the idiosyncratic risk of the price of his own home, beyond the systematic risk of the housing market.

The second use for the Housing Option is in the context of a mortgage restructuring. The Housing Option can be used in conjunction with other available instruments, such as a new first mortgage and a silent second (a lien on the property with principal and interest deferred until sale) in order to make the restructuring more appealing to both the lender and borrower, and to align post-restructuring incentives.

Under the Housing Option, the homeowner writes the option, and lives in the home. This is similar to a covered call strategy and its appropriateness should probably be judged in that light. If the homeowner believes that house prices won't have a pronounced upward trend while he is likely to live in the house, then the Housing Option may be beneficial strictly in financial terms. Of course, there may be additional welfare benefits.

The Housing Option differs from a covered call strategy in several important respects. The most important aspect is that the option does not exist until the homeowner sells the house. The date of a "selling the house" event is unlikely to be known in advance, and the hazard rate for this event may be a function of both time and the house price. In the simplest case, where the hazard rate is constant, the net result is to further discount the payoff, over and above the time value of money.

The remainder of this paper is structured as follows. In section 2, I briefly review the literature on housing exposure in a financial planning context, including products for hedging house price risk, the welfare benefits of hedging that accrue to the homeowner, and some differences between housing market prices and equity prices that need to be considered when thinking about hedging. In Section 3, I describe alternative approaches to mortgage restructuring, as well as the advantages of incorporating the Housing Option into the restructuring strategy. Section 4 presents a high-level overview of the products currently available and their key characteristics. In Section 5, I evaluate the pricing for the Housing Option. The pricing model is detailed in an Appendix. Section 6 concludes.

2 Housing Price Risk

2.1 Products that mitigate housing price risk

Residential real estate risk is a topic of theoretical and practical importance. For many families, housing investment is often much larger than investment in the stock market, with median investments of the US household in a ratio of 4:1 (house/investments, see [Dav02]), and the \$21.6 trillion US housing market is larger than the \$17.0 trillion stock market[1]. A number of mechanisms for mitigating housing price risk have been proposed, promising both to improve the operations of the markets and increase individual family welfare.

Risk can be reduced by diversifying, buying insurance against the risk, or hedging the risk. Diversification is generally not available to the homeowner as a way to reduce house price risk. By contrast with insurance, where the homeowner pays another party to assume the risk, hedging involves entering into an investment with risk characteristics opposite to the one you want to reduce, e.g. a long-short portfolio in the stock markets.

Home equity insurance was the first of these available to homeowners. The history of the development of home equity insurance is reviewed in Caplin *et. al.* [CGH$^+$03], where they describe in detail an index-based home equity insurance program that began operation in Syracuse, New York in 2002. Briefly, the roots of the idea go back to at least 1925 and the passage of Civil Code §453hh in California. This code first regulated land value insurance, and land value insurance was then codified into the California Insurance code in 1935.

The Department of Defense ran the first actual home equity insurance program in the U.S. starting in 1966, under the “Demonstration Cities and Metropolitan Development Act Of 1966.” This program protects both members of the military and civilian contractors from loss in home value resulting from the closing of a nearby military installation.

Marcus and Taussig [MT70] and Yarmolinsky [Yar71] were the first to propose general home equity insurance programs. In both proposals, the payout to the purchaser was based on the difference between the insured value and the actual sale price of the home, by contrast with a payout based on a broader index. Marcus and Taussig believed that this insurance could be publicly financed, while Yarmolinsky suggested that policies should be written by commercial insurers and then re-insured by the government. In 1978, Oak Park, Illinois launched a program similar to the one described by Yarmolinsky, and it became the model for home equity insurance programs.

In 1993, Case, Shiller, and Weiss [CSW96] argued that index-based futures and options that are driven by region-specific movements in house prices would be of benefit in hedging house price risk, not only to mortgage writers, who implicitly write put options on real estate assets, but also to homeowners, the largest bearers of residential real estate price risk. They point out that homeowners can be highly leveraged and are often undiversified.

Homeowners are also unsophisticated and unlikely to use the derivative markets directly, preferring rather risk-management services offered by retailers if they were marketed appropriately. Potential retailers for risk management products and services are discussed in [SW99]. In 2006 the Chicago Mercantile Exchange

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[1] CME Housing Futures and Options, Chicago Mercantile Exchange, May 2006

introduced trading in housing futures for 10 US cities, in addition to futures on a national house price index, using the Case-Shiller indices.

One of the reasons housing is a critical topic for personal financial planning is that owning a house is itself a hedge against the risk entailed by exposure to fluctuations in the cost of housing, which would otherwise need to be obtained in the rental market [WP05]. Owning a house is also a hedge against variations in human capital / labor income, and Davidoff [Dav02] suggests that many homeowners undertake financial investments in a riskier position than usually thought, explaining the appeal of proposals for market or tax-based risk sharing in housing prices.

2.2 Welfare benefits of housing risk reduction

Welfare analysis is an economic analysis that incorporates a notion of personal preference and optimal choice among alternatives, suitably time-discounted to capture personal preference over time. Even before the advent of trading in housing futures, the welfare benefits of a housing futures market were studied. Englund *et. al.* [EWG02] looked at a large set of empirical data on housing in Stockholm and suggest a real benefit from the presence of products that would allow homeowners to hedge their lumpy investments in housing. They conclude that the value is large, particularly for poorer homeowners.

Recent papers have looked at hedging with housing index futures, reflecting the recent launch of residential real estate futures markets. As an example, Voicu [Voi07a] looks at the combined hedging requirements derived from a household's consumption and investment needs. This analysis considers both homeowners looking to trade housing up or down, as well as those with exposure to short term housing price risk when considering a move to a different city.

De Jong *et. al.* [DJDVH08] is the most comprehensive analysis so far, and most useful for this paper. They evaluate the economic benefits of having access to housing futures for home-owning investors, using a model for the portfolio choice between stocks, bonds of various maturity, different mortgage types, and housing futures. They compare the welfare gains of housing futures with the economic benefits of two other important housing-related portfolio decisions: (i) incorporating the housing exposure in financial portfolio choice and (ii) mortgage choice.

The analysis suggests that unless the large idiosyncratic house price risk can be hedged, the portfolio implications and welfare improvements of the housing futures are small. The main reason is that idiosyncratic house price risk cannot be hedged using futures written on a city-level house price index. While these contracts provide exposure to housing price fluctuations on the city-wide level, significant deviations are possible at the neighborhood, block, or individual house level. Presumably, a sufficiently diversified portfolio of individual homes could be hedged with these futures, but even that may prove difficult in practice.

If this large idiosyncratic risk could be hedged however, the welfare improvements would be quite large. Note that because the underlying is a single house, it is exactly the idiosyncratic risk that the Housing Option provides a hedge for. Moreover, the results of De Jong *et. al.* suggest that homeowners near retirement, who expect to stay in their home for 5-10 years before down sizing, would have large welfare gains if they could hedge house price risk.

3 Mortgage Restructuring

3.1 Problem and existing solutions

The drop in housing prices has had disastrous effects for both mortgage lenders and borrowers. This drop has been so severe that many mortgages are now underwater (i.e. the remaining principal on the mortgage greatly exceeds the current market value of the home). This leaves the homeowner with little incentive to continue making his mortgage payments, since he would be in effect paying an above-market amount for the home, which provides a strong incentive to default. From the borrower's standpoint, this situation is clearly undesirable, since by walking away he loses his investment and leaves his goal of homeownership unrealized.

Furthermore, if the homeowner walks away, the lender is left to recover any remaining value in the property. Retail banks are in the business of taking deposits and making loans, and are not real estate investment or holding companies. Therefore, they must find a way to realize the value of the home, often at significant cost. Current estimates place foreclosure costs at approximately 30-60% of outstanding loan value[2]. Therefore, this situation his highly undesirable for the bank as well.

To get a sense of the total wealth destroyed in the United States by the foreclosure process, mortgage debt outstanding increases by approximately $60 billion on an annualized basis[3]. As of the third quarter of 2008, the overall foreclosure rate was around 3% of all loans outstanding, with another 7% identified as delinquent[4]. Assuming that foreclosure costs are 30% of this outstanding loan value (the low end of the range identified above), that means that at least $54 million dollars of wealth is being destroyed by the foreclosure process annually.

The current solutions employed most frequently for restructuring mortgages include the new first mortgage and the silent second (a lien on the property with principal and interest deferred until sale)[O'B08]. These solutions leave much to be desired. If the sum of the values of the new mortgage and silent second exceed the value of the property, the mortgage is still underwater, and re-default is highly probable. However, if the sum of the values is below the market value of the property, the homeowner can sell and realize the newly-created positive equity immediately. In order for the restructuring to be successful, the interests of the lender and borrower must be aligned.

3.2 Advantages of the Housing Option

Combining the Housing Option with either a reduced first mortgage or a combination of a reduced first mortgage and a silent second would mitigate many of these concerns. Assuming that the sum of the values of the first mortgage and silent second is below the current market value of the home, the homeowner would receive a portion of the newly-created equity and any future appreciation. Considering this in conjunction with the benefit derived from living in the home, it is clear that this arrangement provides a strong incentive against re-default. In addition, if structured properly, the bank would receive a safer loan valued at a higher percentage of the home's value than could be realized through foreclosure, in addition to participation in the home's future appreciation. This arrangement is thus mutually beneficial to both borrower and lender. The

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[2] A Report on Lenders' Cost of Foreclosure from the Mortgage Bankers Association, Mortgage Bankers Association, May 2008

[3] Press Release - Debt Outstanding, Mortgage Bankers Association, December 2008

[4] Press Release - NDS, Mortgage Bankers Association, December 2008

risk that the homeowner could sell immediately to realize the equity created is still a concern, but this risk is addressed through certain features of the Housing Option.

4 Existing Housing Option Contracts

4.1 REX Agreement

Rex & Co. offers the "REX Agreement," (`http://www.rexagreement.com/`) a contract between a homeowner and Rex & Co. whereby the homeowner grants a fractional call option on the value of the principal residence in consideration for an initial payment from Rex & Co. The REX Agreement is long-dated, initially for a ten-year term, but renewable by Rex & Co. at each ten-year contract anniversary for an additional ten-year term and for additional consideration. The maximum term of the REX Agreement is 50 years.

In addition to the exercise date being determined by the option writer (the homeowner), entering into the REX Agreement differs from a covered call in another respect. If the call option does not come into being in the first ten-year term (i.e. no "selling the house" event has happened), then Rex & Co. holds another option that exists on the ten-year anniversary date - the option to renew the REX Agreement for an additional ten-year term on the same terms[5]. In the case where the REX Agreement lasts for 50 years, there is no option to renew at the fifty-year anniversary, but instead the call option can be executed by Rex & Co., forcing the sale of the house (alternatively, the homeowner has the right to buy back the option).

Thus the homeowner really writes two (fractional) options when entering into the REX Agreement: a call option on the house value, and a call option on the REX Agreement. The former comes into being if the house is sold within ten years; the latter comes into being if the house is not sold within ten years.

For the original option, the strike price is floating, increasing over the first five years such that if the homeowner sells within the first five years, Rex & Co. receives a larger amount than it would if the home were sold at the same price after five years. This serves to discourage early sale.

There are other events that bring the REX Agreement option into effect, including homeowner default on mortgage payments and default on other covenants in the REX Agreement governing property maintenance, payment of real estate tax liabilities, and keeping enough property and casualty insurance in force. On the other hand, the REX agreement will adjust the strike price upwards to account for home improvements made during the term of the contract.

This structure of the REX Agreement is most suitable to responsible homeowners looking to hedge against idiosyncratic housing price movements.

4.2 HOPE for Homeowners (H4H)

The Federal Housing Authority (FHA) offers "HOPE for Homeowners," (`http://portal.hud.gov/portal/page?_pageid=73,7601299&_dad=portal&_schema=PORTAL`) a contract between a homeowner and the FHA whereby the homeowner grants a fractional call option on the value of his principal residence in return for a

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[5] That is, if it hasn't been exercised, the option is renewable by the holder on each 10th anniversary of the contract date at the same terms, up to a maximum of 50 years, depending on the jurisdiction governing the contract.

reduced first mortgage, a portion of the initial equity created by the mortgage reduction, and a share of the future appreciation of the home. The maximum term of the H4H Agreement is not specified in the public literature, but is presumably at maximum the lifespan of the homeowner.

The structure of HOPE for Homeowners is simpler than the REX Agreement. First, the fraction of future home price appreciation is fixed at 50%. In addition, the length of the H4H Agreement is simply the time until sale. The H4H Agreement is in existence as long as the homeowner owns the house, and accordingly no extension options are necessary.

As opposed to the floating strike incorporated in the REX Agreement, H4H considers the initial equity created by the reduced mortgage separately from any future appreciation. The homeowner receives an increasing portion of this initial equity for each year after initiation that sale does not occur. Like the floating strike, this provision serves to discourage early sale.

The structure of H4H is most suitable to delinquent homeowners in default on their mortgages.

4.3 Home equity fractional interest security (HEFI)

Home Equity Securities LLC (HES) offers the *Home Equity Fractional Interest security* ("HEFI") (`http://www.homeequitysales.com/`), a contract between a homeowner and HES whereby the homeowner grants a fractional call option on the value of his principal residence. The HEFI is a flexible solution, and incorporates features of both contracts discussed above. In the standard version, the homeowner receives a payment similar to that of the REX Agreement. In the foreclosure mitigation version, the homeowner receives a reduction in his first mortgage and a portion of the initial equity created by the mortgage reduction. The homeowner shares in the future appreciation of the home in both versions. The HEFI remains in force until the home is sold or until the homeowner dies.

The structure of the HEFI is simpler than the REX Agreement, and is similar to H4H. Like H4H, the length of the HEFI is simply the time until sale. The HEFI never expires, and accordingly no extension options are necessary.

The foreclosure mitigation version of the HEFI includes a provision whereby the homeowner's interest gradually increases from 20% to its final value of 60% over the first five years of the HEFI.

Although the default participation level for HES is 40%, other versions of the HEFI with varying participation levels are available.

The flexible structure of the HEFI allows for variations aimed at either providing responsible homeowners with a hedge against idiosyncratic housing price movements or providing an attractive restructuring option for delinquent homeowners in default.

4.4 Other products

EquityKey (`http://www.equitykey.com/`) markets a Housing Option exclusively to homeowners aged 65 and older. From their website, their option appears similar to the REX Agreement, but with participation alternatives limited to 50% and 100%. For the 50% participation rate, the marketing materials reference an option premium of 10-15% of the spot price, which is not inconsistent with the REX Agreement. EquityKey also appears to have a higher strike price on their option compared to REX, but requires a life insurance contract on the owners.

The REX Agreement has also been marketed by Grander Financial (`http://www.granderfinancial.com/`) as “My Equity Freedom.”

5 Pricing a Housing Option

5.1 The housing price process

In order to develop a pricing framework for the Housing Option, house prices must be considered, i.e. the prices of individual houses, and their relation to an index of house prices, whether local or national. By contrast with the equities markets, there is substantial persistence in real estate prices over time, housing price changes in one year predicting up to one half the price change the following year (see Case & Shiller [CS89], one of the most influential papers on home prices [ILC06]).

The Case & Shiller article was based on a previous paper by Case and Shiller, in which they proposed and applied the weighted repeat-sales method for constructing real estate indices. The repeat-sales method considers all properties that sold more than once, thus keeping their characteristics constant, and futures markets trading on the Chicago Mercantile Exchange since 2006 are based on the Case-Shiller Index. The Office of Federal Housing Enterprise Oversight (OFHEO) reports similar repeat-sale home price indices for all US metropolitan areas [Cal96]; repeat indices have also been constructed at zip-code level.

These indices in turn allow researchers to analyze the real estate markets in greater depth, and there are interesting applications of the repeat sales method. For example, a study of long-term historical repeat index for a European city (350 years) showed decreasing trends in the house price index over extended periods.

There is a now large literature that documents the forecastability of real estate markets [SW99] . Among these is a study that suggests it takes several years for markets to clear [Mal99].

It is this long-term persistence in prices, and the price dynamics that it implies, that makes pricing house price derivative contracts distinctly different from pricing similar contracts based on equity prices. I look at alternative dynamic models for the price of the underlying in the next section.

For a homeowner thinking of selling within 5-10 years then, house price persistence, particularly in a downward trend as we are currently experiencing, is a risk that might be hedged against, and housing futures contracts based on the index are one way to do that. In addition, it has been pointed out [Voi07b] that home price indices based on physical transactions might overstate an investor’s ability to sell during a downturn. The investor would need to keep the property on the market for an excessively long time to sell at index prices. On the other hand the noise in individual housing prices is large relative to the standard deviation of changes in the index, which makes futures contracts on the index a poor hedge for the individual home price in any case. So what’s missing is a way to hedge the risk on the price of an individual house. This is one of the uses for the Housing Option.

5.2 Exit rates

The Housing Option only comes into existence when the homeowner sells the house and exits the option contract. Thus a critical pricing parameter is how long the holder can expect to wait until he realizes the contract payout, and this waiting time is a function of the probability of sale per unit time, also known as the hazard rate.

There are several empirical studies of homeowner exit (sale) events that can be used as a basis to estimate the hazard rate. Certain studies look at duration of occupancy (time between moving into and out of a residence), such as Hempel and Ayal [HA77] and Soberon-Ferrer and Dardis [SFD91]. Other studies, such as Harsman and Quigley [HQ91], are focused on the similar idea of average age of the current residence (time since moving into current residence). Anily et al. [SA99] parameterize the distribution of total residence duration of a household at the same residence given data on the age of residency in a particular subgroup, using housing survey data published by the Bureau of Census.

There are a number of distributions commonly used for this type of process. Over a short enough time period, all are exponential, i.e. the waiting time decreases exponentially and the hazard rate is constant. Different distributions arise when the hazards are non-constant, and so the choice of distribution can sometimes be informed by a hypothesis about how the hazard rate changes over time. The hypothesized distribution should then be tested with reference to the data, i.e. statistical tests can be used to discriminate among candidate distributions.

In the absence of any other information, the simplest assumption is that the hazard rate is constant, and independent of the time, the underlying price, or any other factor. If there are data available, the maximum likelihood estimate for the hazard rate is the number of first events divided by the total time at risk, or in this context, the number of house sales divided by the sum of the times that the owners occupied the houses from purchase to sale.

5.3 Model framework

My goal in building a way to price the Housing Option is to find a straightforward approach based on standard option-pricing building blocks. A high-level discussion of the model framework follows. The details of the underlying mathematics are provided in the appendix.

The key choice here is deciding how to represent house prices. The OFHEO indices come with enough information (see [Cal96] for the methodology and formula) that you could simulate a time series for house prices and then use a Monte Carlo estimate of the future house price as a basis for estimating the value of the Housing Option. This would probably be the preferred method for the buyer of the option, but the seller really only needs some comfort the that the price is “fair” in some sense on the contract date. Since the seller (homeowner) decides when the buyer’s option exists (by deciding to sell the house) he has more factors under consideration that could effect his welfare than just the money he received in the past for the option.

This situation is similar to what happens with an American option contract on an equity, where the buyer and seller only need to agree on the price on the contract date. The seller must price the option under the assumption that the buyer exercises the option at a time that minimizes the seller’s profit. However, the buyer is under no obligation to do so, and may exercise sub-optimally for his own reasons. For example, a homeowner writing the Housing Option is not required to exercise at an economic-optimal time; he might prefer less house and more secure retirement income to the potential for a house price increase. Indeed, the model described below assumes that the time of exercise is a random variable, independent of the house price.

For the purpose of this analysis I've chosen to model house prices as a fractional Brownian motion ("fBm"), a one-parameter extension to the standard Brownian motion. The parameter is known as the Hurst parameter (after Hurst [MN68]) and can take values in the range [0,1]. Standard Brownian motion has Hurst parameter 0.5, and for Hurst parameters in the range (0.5,1] the fractional Brownian motion is smoother than standard Brownian motion and exhibits long-range persistence.

There is an extensive literature on the use of fBm in finance,[Øks03, MN68, HØUZ96, HØ03, GN96, EvdH03, BØSW03, BH05], and pricing formulas for derivative contracts that correspond to the Black-Scholes formulas [BS73] can be used.

The Hu & Oksendal [HØ03] formula derivation employed in this paper is based on unconditional expectations, and therefore does not take into account the price process history. Rostek [Ros09] develops an approach based on conditional expectations, resulting in a slight modification to the Hu & Oksendal formula.

The main parameters of the model of the Housing Option are as follows:

parameter	description
H_t	house price at time t
$\mathcal{H}$	Hurst parameter
σ	house price volatility
r	risk-free interest rate (nominal)
$h(H_t, t)$	house sale hazard rate
q	rental yield
τ	random time of house sale
ρ	fraction granted to the buyer under the Housing Option
D_0	initial payment to homeowner
D_n	payments to homeowner if renewal is exercised

It is important to note that a rental yield must be included in the model since, while living in the house, the homeowner has the benefit of the equivalent rent payments, while the holder of the Housing Option does not. Rental yield is a standard aspect of economic thinking in respect of house prices. In the general language of financial assets, the house pays a dividend to its owner that does not accrue to the holder of the Housing Option.

The main terms of the Housing Option were described in Section 4. The call option can be valued as a European option on the house price (which is dividend paying, as noted) that comes into existence at the random time representing the time of the house price sale. As such, the call can be priced as the expected value of the payoff under the risk-neutral measure, the payoff occurring at the random time.

Denote the value of this call option by $C(r, q, \sigma, \tau, H_0, D_0, \rho, \mathcal{H})$. As mentioned, in essence this is a European option with expiry τ, the only unusual feature being the assumption that the underlying returns have fractional Brownian dynamics with Hurst parameter $\mathcal{H} \in [0.5, 1]$ (standard Brownian dynamics have Hurst parameter $\mathcal{H} = 0.5$).

The random time of house sale can be thought of as the first jump time of a point process with random intensity $h(H_t, t)$, also known as the hazard rate. I’ve expressed the hazard rate as a function of the both the house price and time to emphasize the opportunity within this model framework (i.e. without a structural, utility-based model of the house sale by the homeowner) to incorporate some aspects of homeowner behavior. For example, the hazard rate could be specified as

$$h(H_t, t) = \begin{cases} \lambda_0 & H_t < H_0 \\ \lambda_0 + \lambda_1 & H_t \geq H_0 \end{cases}$$

such that the homeowner is more likely to sell if the house price is above the price at the contract date than if the price is less. See Carr & Linetsky [CL00] for a discussion of these sorts of models and examples of analytically tractable price dependent hazard rates.

For the base case, assume that the hazard rate governing the house sale is constant with $h(H_t, t) = \lambda$, and so the value of the option on the sale of the house is the expected value of the call option over the possible house sale times[6]. In the initial 10-year period this is [see the Appendix]

$$C_{[0,10)} = \int_0^{10} \lambda e^{-\lambda\tau} C(r, q, \sigma, \tau, H_0, D_0, \rho, \mathcal{H}) d\tau$$

For the REX Agreement, Rex & Co. has the option to renew the REX Agreement at the end of the first 10-year period and so, at the contract date, the homeowner also grants Rex & Co. an option on an option. This second option is referred to as a compound option and only comes into effect if the house is not sold.

Valuation of compound options is discussed in Geske [Ges79], and this one can be valued in a straightforward manner using a European call on the option-to-continue, conditional on the house not being sold before the expiration date. This second option is only exercised if the value of the underlying (another 10 years under the same terms) is greater than the renewal fee D_n. The renewal fee is relatively small, equal to \$1,000 plus $0.005 \times \rho H_0$.

The probability that the house is not sold at the end of the first ten year period is $1 - \int_0^{10} \lambda e^{-\lambda\tau} d\tau = e^{-\lambda 10}$ and so the hazard rate in effect discounts the contribution of the value of the compound option to the total value of the REX Agreement.

5.4 The “fair’ value of the REX Agreement

The following is a high level discussion of the pricing of the REX Agreement. Although the exact prices will differ, the pricing methodologies for the H4H and HEFI are quite similar, and have the same general interpretation. Therefore, the results of this section apply to the Housing Option in general, and explicit discussions of the pricing of the H4H and HEFI are omitted to avoid redundancy. The REX Agreement was chosen because Rex & Co. provides the most detailed information on the product features, allowing for the most accurate model parameterization. All figures stated below are for a two-period REX Agreement, which provides a useful approximation for the five-period REX Agreement for a reasonably high sale intensity.

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[6] I am ignoring for the moment the fact that the REX Agreement reduces the strike price if the house is sold in the first five years of the REX Agreement.

Given the contract date value of the house (H_0) and the fraction to be shared (ρ), then in consideration for entering into the REX Agreement, Rex & Co. pays the homeowner a fee (D_0) at closing.

If the initial house price is H_0 and the house is later sold for a price of H_t, Rex & Co. gets the payout

$$\rho \max\left(H_t - \left(H_0 - \frac{D_0}{\rho}\right), 0\right)$$

paid from the proceeds, and the homeowner keeps the balance. It should be noted that, in order to discourage early sale, the strike price for the option is initially reduced by 25% of the advance payment D_0, and increases by $.05D_0$ each year until it reaches its final value. This is clearly advantageous to Rex & Co., and is referred to as the "early exit cost" in the Rex & Co. literature.

For a homeowner thinking of selling within a moderate time frame (5-10 years) and who is concerned with the risk of a decrease in house prices while willing to gamble that house prices don't see a dramatic increase over the period, the REX Agreement may be attractive.

The fee paid to the homeowner (D_0) should be equal to the "fair" price of the options sold to Rex & Co. by the homeowner. For the purpose of estimating this theoretical "fair" price, I've largely used the parameter values based on the paper by De Jong & Driessen [DJDVH08]. In particular[7], US house price volatility (idiosyncratic plus systematic) is assumed to be 12%, while the risk-free rate (real plus inflation) is assumed to be 5%.

A key parameter in the pricing model used here is the equivalent rental yield. While De Jong & Driessen derive implied rents that are less than 1% (0.067%), Voicu [Voi07a] finds much higher imputed rents (> 8%) based on the futures prices. I've used rental yields in-between these two extremes here, but the prices are quite sensitive to the assumed rental yield.

Using the methodology outlined in the Appendix, the calculated value of the REX Agreement[8] as a percentage of initial home value is shown in the table below, for different values of the implied rents (rental yield q), the average time between house sales (1/hazard rate=$1/\lambda$ in years) and the persistence of house model prices (Hurst parameter $\mathcal{H}$)

	$1/\lambda = 4$			$1/\lambda = 8$			$1/\lambda = 12$		
	$q = 2.5\%$	$q = 3.5\%$	$q = 4.5\%$	$q = 2.5\%$	$q = 3.5\%$	$q = 4.5\%$	$q = 2.5\%$	$q = 3.5\%$	$q = 4.5\%$
$\mathcal{H} = 0.7$	15.37%	14.03%	12.82%	14.98%	13.14%	11.56%	14.42%	12.35%	10.63%
$\mathcal{H} = 0.6$	15.22%	13.80%	12.50%	14.78%	12.78%	11.03%	14.24%	11.94%	10.00%
$\mathcal{H} = 0.5$	15.08%	13.61%	12.27%	14.57%	12.47%	10.64%	14.00%	11.58%	9.53%

As should be expected from the Black-Scholes Greeks for a vanilla call option, the sensitivity of the Housing Option price to the riskless interest rate (rho) is positive. This reflects the opportunity cost savings compared with taking a similar position through direct investment in the underlying, and the rate of interest that can

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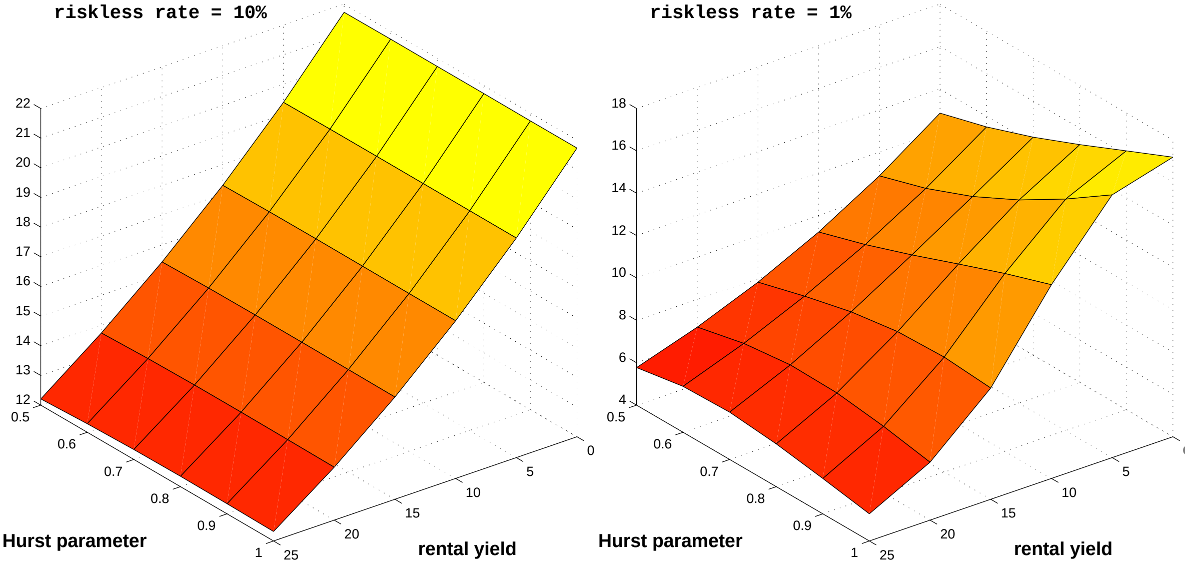
[7] In De Jong & Driessen, parameter values for the real interest, expected inflation and unexpected inflation rate are calibrated to quarterly US data on nominal interest rates and inflation from 1973 (Q3) to 2003 (Q4). The house price dynamics are calibrated to OFHEO repeated-sales index data for 10 US cities from 1980 (Q2) to 2003 (Q4). House price volatility depends on the city and should probably be estimated separately in practice.

[8] Given pricing parameters $r = 0.05$, $\sigma = 0.12$, $S_0 = 750,000$, $D_0 = 88,867$, $D_n = 2875$, and $\rho = 0.5$, based on a quoted deal closed by Rex & Co.

Figure 1: Price as a function of Hurst parameter and rental yield

The two charts below show the estimated REX Agreement value as a percentage of initial home value, assuming two different riskless rates, given pricing parameters $\sigma = 15\%$, $\lambda = \frac{1}{8}$, $S_0 = \$1,000,000$ and $\rho = 40\%$. At each riskless rate, the variation in price is shown as a function of the Hurst parameter, and the rental yield.



be earned on the amount saved. In general, the Housing Option price is a concave function of the riskless rate, although some exceptions arise with extreme values of other parameters. Furthermore, high values of the riskless rate tend to dampen the effects of the volatility and Hurst parameter (see Figure 1) on the value of the Housing Option, since the discounting outweighs the increased possibility of moving further into the money.

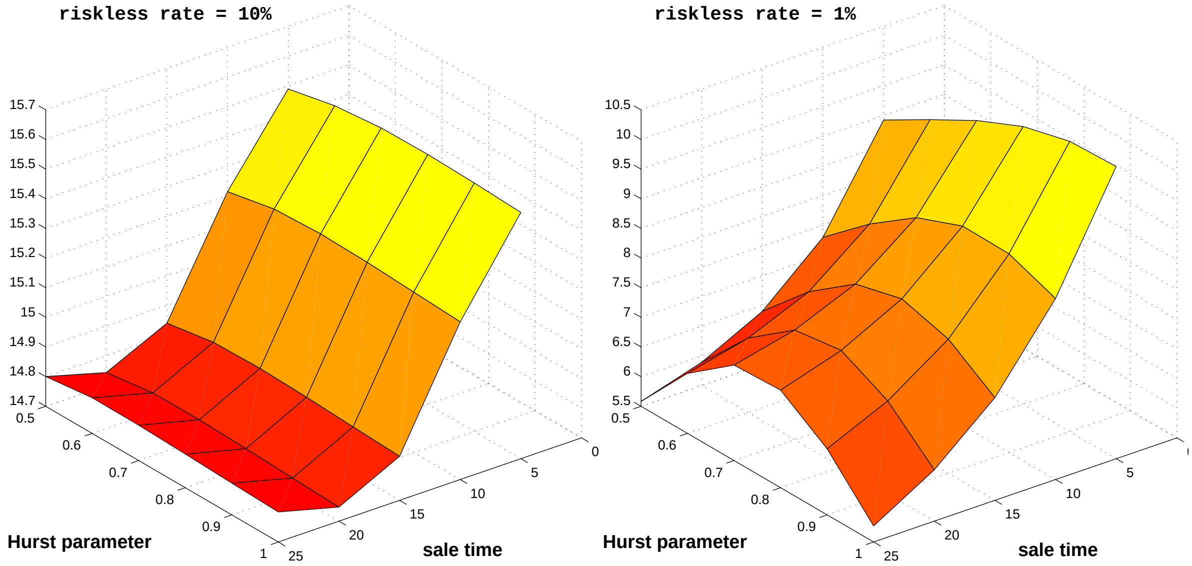
The sensitivity of the Housing Option price to the rental yield (phi) is negative, as expected given the sensitivity of a vanilla call option. The rental yield represents a benefit which accrues to the homeowner (the option writer), but not the option holder, thereby decreasing the option value. The Housing Option value is typically a convex function of the rental yield, with some exceptions arising from extreme values of the other parameters. There is a very sensitive relationship between the rental yield and the average time between sale events, with the sign of the sensitivity to time between sales actually changing depending on the level of the rental yield. In general, the rental yield has an opposite effect to that of the riskless interest rate, and low values of the rental yield tend to dampen the effects of the volatility and Hurst parameter on the Housing Option value accordingly.

The sensitivity of the Housing Option price to the volatility (vega) is positive, as should be expected from vanilla calls. The increased volatility results in a greater chance for the option to move further into the money by expiration. The price is generally a concave function of volatility, with some exceptions arising from extreme values of the other parameters. Due to its appearance in the denominator of the pricing formula, a value of zero volatility results in an undefined value for the Housing Option, although negligible volatility results in a low time value for the option. Furthermore, high volatility increases the effect of the Hurst parameter on the Housing Option value.

The sensitivity of the Housing Option price to the Hurst parameter is relatively minor, compared with some of the other parameters. The increased persistence in the housing price process has a very similar effect

Figure 2: Price as a function of Hurst parameter and expected time to sale

The two charts below show the estimated REX Agreement value as a percentage of initial home value, assuming two different riskless rates, given pricing parameters $q = 3\%$, $\sigma = 15\%$, $S_0 = \$1,000,000$ and $\rho = 40\%$. At each riskless rate, the variation in price is shown as a function of the Hurst parameter, and the expected time to sale.



to a volatility increase. The price is generally a concave and increasing function of the Hurst parameter, although for certain extreme values of the other parameters (such as very low values of the riskless rate), this sensitivity can actually become negative (see Figure 2). The Housing Option price becomes most sensitive to the Hurst parameter when the riskless rate is low, the rental yield is high, the volatility is high, or the average time between sales is high. The low riskless rate (or high rental yield) effectively results in less discounting, accentuating the effect of the diffusion. Both increased volatility and high average time between sales result in wider diffusion of prices, and therefore an increased possibility of moving into the money. High values of the Hurst parameter can result in kinks in the graphs of Housing Option price vs. many of the other parameters.

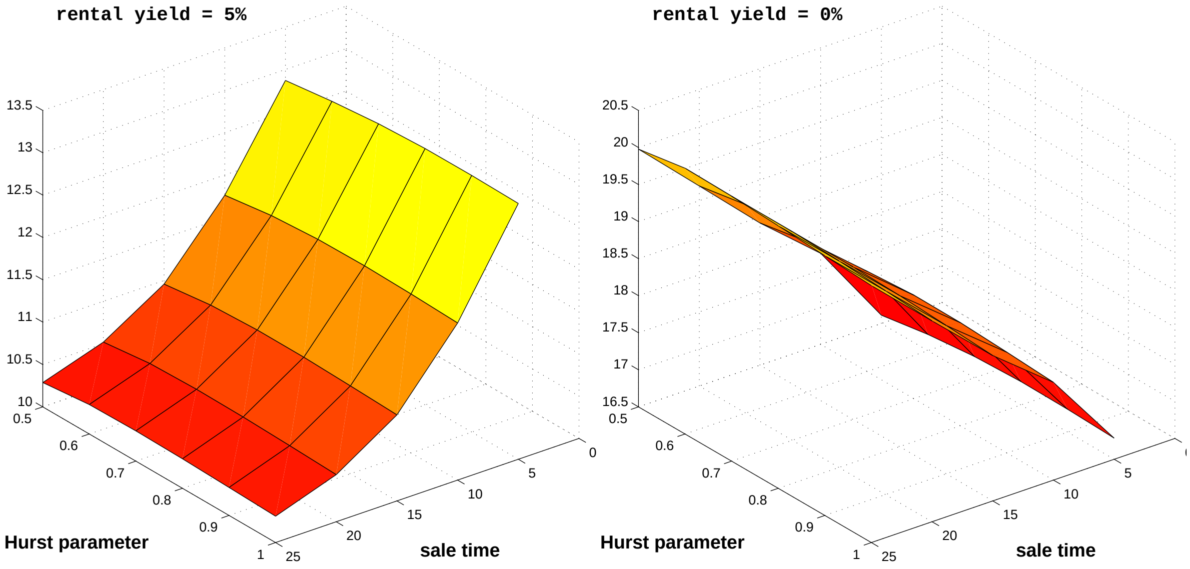
The sensitivity of the Housing Option price to the average time between sales is generally negative, reflecting the value of the rental yield which does not accrue to the option holder. For this reason, the sensitivity to average time between sales actually becomes positive for low values of the rental yield (see Figure 3). The price is generally a convex function of the average time between sales, although exceptions do arise for extreme values of the other parameters. High average time between sales increases the sensitivity to volatility, since the housing price has a longer time to diffuse. Since a zero average time between sales results in an effective maturity of zero for the option, and the maturity appears in the denominator of the pricing function, this results in an undefined value for the Housing Option price. However, negligible values for the average time between sales result in high prices, due to the almost immediate payment.

5.5 The “fair’ value of the H4H Agreement

The pricing of the H4H Agreement is very similar to that of the REX Agreement, with a few notable differences. First, the option granted to the FHA remains in force until the sale of the home, and therefore has a theoretically infinite maturity. Extension options are not necessary, so the compound option component

Figure 3: Price as a function of Hurst parameter and expected time to sale

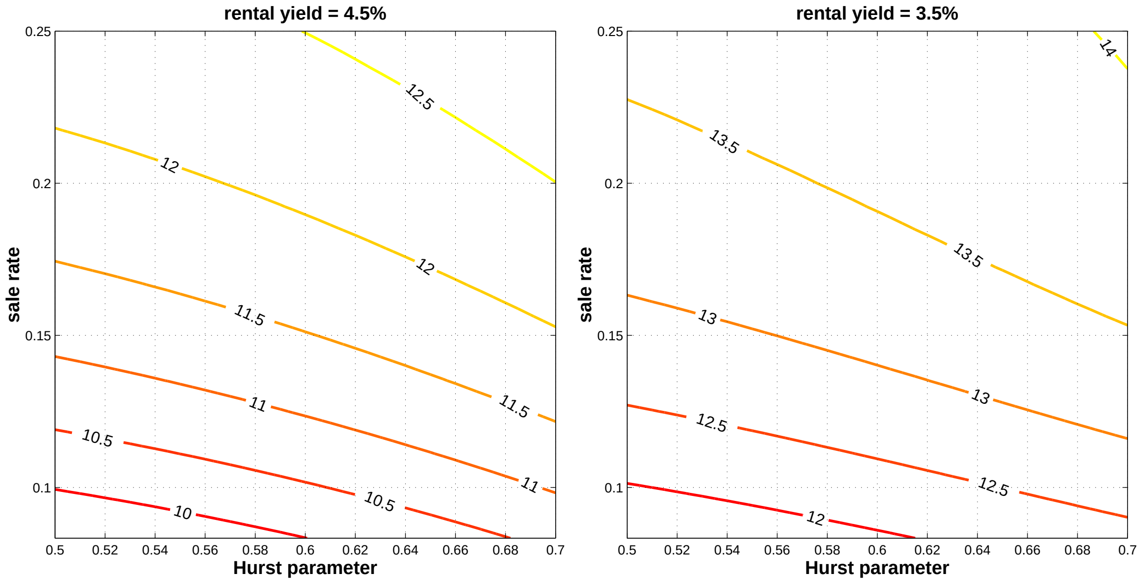
The two charts below show the estimated REX Agreement value as a percentage of initial home value, assuming two different rental yields, given pricing parameters $r = 5\%$, $\sigma = 15\%$, $S_0 = \$1,000,000$ and $\rho = 40\%$. At each riskless rate, the variation in price is shown as a function of the Hurst parameter, and the expected time to sale.


`rental yield = 5%`

`rental yield = 0%`

Figure 4: Value contour plot for Hurst parameter and expected time to sale

The two charts below show contour plots for the estimated REX Agreement value, assuming two different rental yields, given pricing parameters $r = 5\%$, $\sigma = 12\%$, $S_0 = \$750,000$, $D_0 = \$88,867$, $D_n = \$2875$, and $\rho = 50\%$, based on a quoted deal closed by Rex & Co. Each line represents the combinations of Hurst parameter and sale rate (the inverse of the expected time to sale) which produce the same value for the REX Agreement, expressed as a percentage of S_0. In this particular case, the payment to the writer at closing was 11.85% of S_0, and the monthly rents corresponding to rental yields of 3.5% and 4.5% are \$2188 and \$2812 respectively.


rental yield = 4.5%

rental yield = 3.5%

of the pricing model can be ignored.

Furthermore, the provision to discourage early sale in the H4H Agreement is different than the REX Agreement. This is a consequence of the different option premium structure. In the REX case, a cash premium is paid directly to the homeowner. In contrast, in the case of the H4H Agreement, the consideration for entering into the contract is in the form of a reduced first mortgage, plus the resulting equity created for the homeowner. If the home is sold immediately, the FHA is entitled to 100% of this newly-created equity. The amount received by the FHA is reduced by 10% annually, until a final 50/50 split is reached.

5.6 The "fair' value of the HEFI Agreement

The pricing of the HEFI Agreement follows a similar procedure to that used for pricing the REX Agreement and H4H, with a few minor differences. First, the option granted to HES remains in force until the sale of the home, and therefore has a theoretically infinite maturity. Extension options are not necessary, so the compound option component of the pricing model can be ignored, as in the H4H pricing.

Furthermore, the provision to discourage early sale in the HEFI Agreement is different than those in either the REX Agreement or H4H. There is no provision to discourage early sale for the standard version of the HEFI, which further simplifies the pricing methodology. In the foreclosure mitigation version, early sale is discouraged with a floating participation level. This is in contrast to the floating strike for the REX Agreement or the varying equity split for H4H, and is accounted for with a slight modification to the pricing methodology.

6 Conclusion

In this paper, I have laid out a framework for pricing options on real estate assets, with a particular focus towards specific products currently offered by government and private-sector entities. I have built upon the previous literature to model the unique characteristics of the Housing Option: writer-determined exercise time and persistent price process of the underlying.

I have also examined the practical applications of the Housing Option. Given the current housing crisis in the United States and abroad, the Housing Option presents a very attractive alternative to the traditional restructuring instruments: the reduced first mortgage and the silent second, reducing the inefficiencies caused by the foreclosure process. Additionally, for responsible homeowners looking to hedge the price risk of their homes, the Housing Option provides an effective hedge over and above that offered through other hedging instruments, such as regional housing price index futures.

The model also provides some comfort to the homeowner considering entering into a Housing Option agreement. The REX Agreement literature states that the premium paid is somewhere between 10-15% of the current home value. This is in line with the prices calculated above. Although data were not available on the pricing of the H4H or HEFI contracts, this would also be a reasonable range for those prices. Of course, any effective premium (in the form of a reduced first mortgage) should be desirable to homeowners in default.

As with any model, it is an imperfect and simplistic representation of reality. However, given the fact that reality cannot be definitively predicted in advance, this model provides a useful way of conceiving and valuing these types of options. I address some of the limitations of the model, and other potential avenues to explore below.

By contract structure, these Housing Options are call options on the value of a individual house that are written by the homeowner. They have at least three characteristics that make them a challenge to value

- the time of exercise is determined by the writer, not the holder,
- the option is very long dated, up to 50 years depending on the state, and
- the price of the underlying is not a Markov process.

That the writer determines the time of exercise is not too difficult to model. The two main modeling choices are between a structural model and a reduced form model. A structural model requires a view on what causes a homeowner to force exercise of the option. The reduced-form model assumes that the time of exercise is unpredictable. Here I chose to sacrifice an economic rationale for the exercise process, in exchange for the mathematical tractability of the reduced form model. Still, a good structural model may provide a more accurate representation of sale events, and would be an interesting extension to explore.

Second, assuming that the price process is stationary is defensible for a 90-day option. For very long-dated options it is not at all realistic. While there are techniques to address this question (e.g. stochastic volatility, regime switching, business cycle models, etc.) Here I assume stationarity, and parameterize the model with estimates of long-term average realized volatility and interest rates. Addressing this limitation of the model through the techniques described above may also result in greater model accuracy.

Finally, house prices demonstrate persistence, and returns are more predictable than they are for stock prices. To model house prices I assume that their returns have the correlation properties of fractional Brownian motion (fBm) and use the fBm option pricing theory of Øksendal . As with the time of exercise, I chose mathematical tractability in preference to a strong economic interpretation for some parts of the pricing theory.

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This paper benefited from detailed and substantive comments by Eric Reiner and Benn Eifert. I would also like to acknowledge the time that John O'Brien generously contributed to reading several drafts, and the information that he provided on these contracts and on real estate markets. Finally, I would like to thank Lou Odette for introducing me to this topic and providing guidance throughout the process.

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Appendix

Valuation Approach: Option with Random Exercise Time

Assume a general contingent claim initiated at time $t = 0$ with a payoff $C\left(S_t, K\right)$ that is some function of the underlying price S_t and a strike price K. The claim pays $C\left(S_t, K\right)$ on the occurrence of an exit event at time $\tau > 0$, i.e. the realized payoff is $C\left(S_\tau, K\right)$. Assume that the price can be determined using standard no-arbitrage arguments for contingent-claim pricing. The analysis that follows is similar to that found in [CL00].

Assume that the underlying price process can be described by a simple geometric random-walk

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

with return μ, volatility σ, and where the risk-free return is r.

The time of exit can be thought of as the time of the first event of a process occurring with a random intensity or instantaneous hazard rate h_t. The hazard rate can be thought of as the probability that the exit event occurs in the next instant of time. The hazard rate is potentially a function of time, of the underlying price, and possibly some other parameters. For simplicity, let $h_t = h(t, S_t)$. The survival process $\bar{F}(t)$ is related to the exit event, and gives the probability that the exit event occurs later than time t (exit time $\tau > t$). Conditional on a given realization of the stock price process the survival function is

$$\bar{F}(t|S_u) = e^{-\int_0^t h(u,S_u)du}$$

and the unconditional survival function is found by taking the risk-neutral expectation (under measure $\mathcal{Q}$) with respect to all price paths, starting at $t = 0$ and initial stock price S_0

$$\bar{F}(t) = \mathbb{E}^{\mathcal{Q}}_{0,\mathrm{S_0}}\left[e^{-\int_0^t h(u,S_u)du}\right]$$

Assuming an expiration date for the contingent claim at time T, the value of the claim at any time $t \in [0,T]$ is

$$V(S_t,t,K,T) = \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau<T\}}C\left(S_\tau,K\right)\right]$$

where $\mathbb{1}_{\{\tau<T\}}$ is an indicator function that is 1 if $t < \tau < T$ and zero otherwise. In this case the expectation is taken over both the random price and the random time. Using the definition for the survival function, the value can be written as

$$V(S_t,t,K,T) = \int_t^T e^{-r(u-t)}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-\int_t^u h(s,S_s)ds}h_u C\left(S_u,K\right)\right]du \tag{2}$$

which says that for all possible ($\int_t^T du$) exit times u greater than t, the risk-neutral expectation of the payoff ($C\left(S_u,K\right)$) is discounted ($e^{-r(u-t)}$), given that the exit hasn't occurred since time t, i.e. has survived ($e^{-\int_t^u h(s,S_s)ds}$) but that it occurs within the increment of time du (i.e. with probability $h_u du$).

Since I've assumed that the underlying follows a geometric Brownian motion per equation (1), then for $u \geq t$ and under the risk-neutral measure

$$S_u = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(u-t)+\sigma\sqrt{u-t}\phi}$$

where ϕ is a unit Normal random variable. If the hazard rate is constant ($h(t,S_t) = \lambda$), equation (2) can be written as

$$\begin{aligned}
V(S_t,t,K,T) &= \int_t^T e^{-r(u-t)}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-\int_t^u h(s,S_s)ds}h_u C\left(S_u,K\right)\right]du \\
&= \int_t^T e^{-(r+\lambda)(u-t)}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[C\left(S_u,K\right)\right]\lambda du \\
&= \int_t^T e^{-\lambda(u-t)}\left[e^{-r(u-t)}\int_{-\infty}^{\infty} C\left(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(u-t)+\sigma\sqrt{u-t}y},K\right)\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy\right]\lambda du
\end{aligned}$$

since $\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$ is the probability density for a unit Normal random variable. The inner integral is the value of a European contingent claim expiring at the (random) time u.

The Feynman-Kac theorem says that the value $V(S_t,t,K,T)$ is also the solution to the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - (r + h(t,S))V + h(t,S)C\left(S_t,K\right) = 0 \tag{3}$$

subject to the appropriate terminal condition.This is similar to the Black-Scholes PDE, and can be solved analytically in some simple cases, or it can be solved numerically.

Compound Options

A compound option is an option whose underlying is another option. Assume a European contingent claim with a payoff $C_2\left(S_{T_2}, K_{T_2}\right)$ that is some function of the underlying price at expiration S_{T_2} and a known strike price K_{T_2}, and which has value $V_2\left(S_t, K_{T_2}\right)$ for $t < T_2$. In addition, assume a European contingent claim initiated at time $t = 0$ and expiring and time $T_1 < T_2$ that pays off $C_1\left(V_2\left(S_{T_1}, K_{T_2}\right), K_{T_1}\right)$, i.e. this second claim's underlying is the value of the first claim. Finally, assume that S_t satisfies equation (1).

From basic contingent claims theory (Martingale approach), the value of the second contract at time $t < T_1$ can be written as

$$\begin{aligned} V(S_t, K_{T_1}) &= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(T_1-t)}C_1\left(V_2\left(S_{T_1}, K_{T_2}\right), K_{T_1}\right)\right] \\ &= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(T_1-t)}C_1\left(\mathbb{E}^{\mathcal{Q}}_{\mathrm{T_1,S_{T_1}}}\left[e^{-r(T_2-T_1)}C_2\left(S_{T_2}, K_{T_2}\right)\right], K_{T_1}\right)\right] \end{aligned} \tag{4}$$

noting that the inner expectation is conditional on S_{T_1}, and which makes clear that $V_1(S_t, K_{T_1})$ depends on the two random variables S_{T_1}, S_{T_2}. The expectation is taken under the risk-neutral measure $\mathcal{Q}$, a change of measure that makes $W^{\mathcal{Q}}_t = W_t + \frac{\mu - r}{\sigma}t$ a standard Brownian motion, given the standard Brownian motion W_t under the physical measure (equation (1)).

Since I've assumed that S_t follows a geometric Brownian motion per equation (1), then under the risk-neutral measure

$$\begin{aligned} S_{T_1} &= S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T_1+\sigma\sqrt{T_1}\phi_1} \\ S_{T_2} &= S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T_2+\sigma\sqrt{T_2}\phi_2} \end{aligned}$$

where ϕ_1, ϕ_2 are correlated unit Normal random variables. The correlations come from the fact that the log-prices are Brownian motions with covariance $\sigma^2 \min\left(T_1, T_2\right)$, and thus the pair ϕ_1, ϕ_2 has correlation matrix (same as the covariance matrix, since these are unit Normals)

$$\Sigma = \begin{pmatrix} 1 & \sqrt{\frac{T_1}{T_2}} \\ \sqrt{\frac{T_1}{T_2}} & 1 \end{pmatrix}\ ;\ |\Sigma| = 1 - \frac{T_1}{T_2}$$

Since the probability density for a pair of correlated unit Normal random variables (x, y) is

$$\frac{e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^{\top}}}{2\pi\sqrt{|\Sigma|}} \tag{5}$$

equation (4) could be written using the density of equation (5) and the expectations could be calculated by integrating directly, as shown below.

Assume both claims are vanilla call options, i.e. $C_1\left(a, b\right) = C_2\left(a, b\right) = \max\left(a - b, 0\right)$ (a "call-on-call"). In this case, for the second contract (first to expire) to be in-the-money at T_1, the value of the second to expire contract $V_2\left(S_{T_1}, K_{T_2}\right)$ has to be above the strike price K_{T_1}. Since this contract is a European call struck at K_{T_2} and expiring at T_2 the value of the contract can be determined as a function of S_{T_1}, and since the contract value is increasing in S_{T_1}, the $S^*_{T_1}$ where $V_2\left(S^*_{T_1}, K_{T_2}\right) = K_{T_1}$ can presumably be found. Thus $C_1\left(V_2\left(S_{T_1}, K_{T_2}\right), K_{T_1}\right)$ will be in-the-money for all

$$\begin{aligned} S_{T_1} &= S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T_1+\sigma\sqrt{T_1}\phi_1} \geq S^*_{T_1} \\ \phi^*_1 &\geq \frac{1}{\sigma\sqrt{T_1}}\left(\ln\left(\frac{S^*_{T_1}}{S_0}\right) - \left(r-\frac{1}{2}\sigma^2\right)T_1\right) \end{aligned}$$

where the second line is just a re-arrangement of the first to get the critical value of the unit Normal variable above which the contract is in-the-money at T_1.

Using this critical value and equation (5), equation (4) can be written as

$$\begin{aligned} V(S_t, K_{T_1}, K_{T_2}) &= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-rT_1}C_1\left(\mathbb{E}^{\mathcal{Q}}_{\mathrm{T_1,S_{T_1}}}\left[e^{-r(T_2-T_1)}C_2\left(S_{T_2}, K_{T_2}\right)\right], K_{T_1}\right)\right] \\ &= e^{-rT_2}\int_{x^*}^{\infty}\int_{-\infty}^{\infty}\left[C_2\left(S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T_2+\sigma\sqrt{T_2}y}, K_{T_2}\right)\right. \\ &\qquad \left. -K_{T_1}e^{r(T_2-T_1)}\right]\frac{e^{-\frac{1}{2}(x,y)\Sigma^{-1}(x,y)^{\top}}}{2\pi\sqrt{|\Sigma|}}dydx \end{aligned} \tag{6}$$

and this can be reduced to the two-period compound option formula found in the literature (e.g. Haug).

More simply, a multi-period call-on-call can be written as a sum of expectations by a multi period extension of equation (4). For example, the three-period call-on-(call-on-call) has value

$$\begin{aligned} V(S_t, K_{T_1}, K_{T_2}, K_{T_3}) &= e^{-T_3}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[S_{T_3}\mathbb{1}_{\{S_{T_3}>K_{T_3}\}}\mathbb{1}_{\{S_{T_2}>S^*_{T_2}\}}\mathbb{1}_{\{S_{T_1}>S^*_{T_1}\}}\right] \\ &\quad -e^{-T_3}K_{T_3}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[\mathbb{1}_{\{S_{T_3}>K_{T_3}\}}\mathbb{1}_{\{S_{T_2}>S^*_{T_2}\}}\mathbb{1}_{\{S_{T_1}>S^*_{T_1}\}}\right] \\ &\quad -e^{-T_2}K_{T_2}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[\mathbb{1}_{\{S_{T_2}>S^*_{T_2}\}}\mathbb{1}_{\{S_{T_1}>S^*_{T_1}\}}\right] \\ &\quad -e^{-T_1}K_{T_1}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[\mathbb{1}_{\{S_{T_1}>S^*_{T_1}\}}\right] \end{aligned}$$

where $S^*_{T_1}$ is the stock price that puts the two-period call-on-call in-the-money at time T_1, and $S^*_{T_2}$ is the stock price that puts the last-period call option in-the-money at time T_2. The last three terms can be calculated from the trivariate, bivariate and univariate cumulative Normal distributions respectively, under the risk-neutral $\mathcal{Q}$-measure.

To calculate the first term in the sum, change the measure per the Radon-Nikodym derivative $d\overline{\mathcal{Q}}/d\mathcal{Q} = e^{-\frac{\sigma^2}{2}T_3-\sigma W^{\mathcal{Q}}_{T_3}}$ so that $S_{T_3} = S_t\left(d\overline{\mathcal{Q}}/d\mathcal{Q}\right)$ and $W^{\overline{\mathcal{Q}}}_t = W^{\mathcal{Q}}_t - \sigma t$ is a standard Brownian motion. Then the first term can be written as

$$\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[S_{T_3}\mathbb{1}_{\{S_{T_3}>K_{T_3}\}}\mathbb{1}_{\{S_{T_2}>S^*_{T_2}\}}\mathbb{1}_{\{S_{T_1}>S^*_{T_1}\}}\right] = S_t\mathbb{E}^{\overline{\mathcal{Q}}}_{\mathrm{t,S_t}}\left[\mathbb{1}_{\{S_{T_3}>K_{T_3}\}}\mathbb{1}_{\{S_{T_2}>S^*_{T_2}\}}\mathbb{1}_{\{S_{T_1}>S^*_{T_1}\}}\right]$$

which can be calculated using a trivariate cumulate Normal distribution, under the $\overline{\mathcal{Q}}$-measure.

Compound Option with Random Exercise Time

If the exercise time is a random variable independent of the price, then the value of any contingent claim can be decomposed into a sum of terms, each corresponding to a period of time, e.g. for $0 < T_1 < T_2$

$$\begin{aligned}
V(S_t, K) &= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau<T_2\}}C\left(S_\tau, K\right)\right] \\
&= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau<T_1\}}C\left(S_\tau, K\right)\right] + \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{T_1<\tau<T_2\}}C\left(S_\tau, K\right)\right] \\
&= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau<T_1\}}C\left(S_\tau, K\right)\right] + P(\tau \geq T_1)e^{-rT_1}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\hat{\tau}-t)}\mathbb{1}_{\{\hat{\tau}<T_2-T_1\}}C\left(S_{\hat{\tau}+T_1}, K\right)\right]
\end{aligned}$$

where I've introduced the random variable $\hat{\tau} = \tau - T_1$ so that the second term has the same form as the first. The first term is the value if the event time satisfies $\tau < T_1$. The second term is the value if the event time satisfies $\hat{\tau} < T_2 - T_1$, given that $\tau \geq T_1$.

This approach can be used to calculate the value if, for example, there are time periods where the contract terms change slightly, e.g. changes in the strike from the first year to the second year, etc.

The two-period REX Agreement has the form of a compound option (with exercise strike K_2 and continuation strike K_1), where the holder has to make a payment at time T_1 if the option is in the money with respect to the period $T_2 - T_1$, and so the critical price S_{T_1} that puts the second period option in-the-money must be found. However, the value function can be written as

$$\begin{aligned}
V(S_t, K_1K_2; T_1, T_2) &= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau<T_1\}}C\left(S_\tau, K_2\right)\right] \\
&\quad + P(\tau \geq T_1)e^{-rT_1}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\hat{\tau}-t)}\mathbb{1}_{\{\hat{\tau}<T_2-T_1\}}\left(C\left(S_{\hat{\tau}+T_1}, K_2\right) - K_1e^{r\hat{\tau}}\right)\right]
\end{aligned}$$

where the new term is the forward value of the strike K_1.

By the same argument, the three-period REX option can be written as (now with $\tilde{\tau} = \tau - T_2$)

$$\begin{aligned}
V(S_t, K_1K_2, K_3; T_1, T_2, T_3) &= \mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tau-t)}\mathbb{1}_{\{\tau<T_1\}}C\left(S_\tau, K_3\right)\right] \\
&\quad + P(\tau \geq T_1)e^{-rT_1}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\hat{\tau}-t)}\mathbb{1}_{\{\hat{\tau}<T_2-T_1\}}\left(C\left(S_{\hat{\tau}+T_1}, K_3\right) - K_1e^{r\hat{\tau}}\right)\right] \\
&\quad + P(\tau \geq T_2)e^{-rT_2}\mathbb{E}^{\mathcal{Q}}_{\mathrm{t,S_t}}\left[e^{-r(\tilde{\tau}-t)}\mathbb{1}_{\{\tilde{\tau}<T_3-T_2\}}\left(C\left(S_{\tilde{\tau}+T_2}, K_3\right) - K_2e^{r\tilde{\tau}} - K_1e^{r(\tilde{\tau}+T_2-T_1)}\right)\right]
\end{aligned}$$

which clearly divides the problem up into chunks, depending on the value of the random time τ. In order to figure out the critical prices at the times T_1, T_2, there are three random variables associated with the price in this case.

Two period Call-on-Call with random exercise.

- In this problem, the first term is just a call option with a random exercise time that expires worthless at the end of the first period. The second term is a compound option.
- The critical price $S^*_{T_1}$ for the compound option is determined with the reasoning used to find the value of the first piece, with the exception that, if $\hat{\tau} > T_2 - T_1$ the option is a regular call option (i.e. does not expire worthless), so when figuring $S^*_{T_1}$ the value of the European call weighted by the probability that $\hat{\tau} > T_2 - T_1$ must be added.

- The last piece is a compound option with a random exercise time $\tau \leq T_2$, so the value function looks like equation (6) except that it must be weighted by (in the constant hazard rate case) $\lambda e^{-\lambda u}$ and integrated over all times u.
- However, since the exercise time determines one of the random variables (the price at exercise), this changes the correlation matrix to

  $$\Sigma_\tau = \begin{pmatrix} 1 & \sqrt{\frac{T_1}{\tau}} \\ \sqrt{\frac{T_1}{\tau}} & 1 \end{pmatrix}; \ |\Sigma_\tau| = 1 - \frac{T_1}{\tau}$$

  where $\tau > T_1$.

Fractional Brownian Motion

Definition

fBm is a natural one-parameter extension of Brownian motion. For Hurst parameter $H \in (0,1)$, the fBm B_t^H is a continuous, centered, Gaussian process with covariance

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \equiv R_H(t,s) \tag{7}$$

and:

1. $B_0^H = 0$,
2. for $s, t \geq 0$, increments of the process are homogeneous, i.e. $B_{t+s}^H - B_s^H$ has the same law as B_t^H,
3. B_t^H is Gaussian, with $\mathbb{E}\left[\left(B_t^H\right)^2\right] = \frac{1}{2}\left(t^{2H} + t^{2H} - |t-t|^{2H}\right) = t^{2H}$, and
4. the trajectories of B_t^H are continuous.

Basic Properties

Self-similarity

Self-similar processes have the same finite-dimensional probability distributions $P\left\{X_{at_0} < x_0, \ldots X_{at_n} < x_n\right\} = P\left\{bX_{t_0} < x_0, \ldots bX_{t_n} < x_n\right\}$ for every choice of $t_0, \ldots t_n \in \mathbb{R}$ and for $\forall a > 0$ and some $b(a) > 0$. For $b = a^{-H}$, B_t^H is self-similar per its definition.

Covariance of intervals and long-range dependence

With $s + h \leq t$ and $t - s = nh$, the covariance between the intervals $B_{t+h}^H - B_t^H$ and $B_{s+h}^H - B_s^H$ can be written as

$$\begin{aligned} \mathbb{E}\left[\left(B_{t+h}^H - B_t^H\right)\left(B_{s+h}^H - B_s^H\right)\right] &= \mathbb{E}\left[B_{t+h}^H B_{s+h}^H - B_t^H B_{s+h}^H - B_s^H B_{t+h}^H + B_t^H B_s^H\right] \\ &= \frac{1}{2}\left(|(t-s)-h|^{2H} + |(-(t-s)-h|^{2H} - |t-s|^{2H}\right) \end{aligned}$$

and so the covariance depends only on the difference $t-s$. Assuming $t-s=nh$ gives the covariance as

$$\rho(n)=\frac{1}{2}h^{2H}\left((n+1)^{2H}+(n-1)^{2H}-2n^{2H}\right) \tag{8}$$

so increments of fBm are positively correlated for $H>\frac{1}{2}$ and negatively correlated for $H<\frac{1}{2}$ (and uncorrelated for $H=\frac{1}{2}$).

Long-range dependence is defined in terms of the auto-covariance function $\rho(n)\equiv \operatorname{cov}(B_t^H B_{t+n}^H)$ if, for some constant c and $\alpha\in(0,1)$

$$\lim_{n\to\infty}\frac{\rho(n)}{cn^{-\alpha}}=1$$

i.e. the covariance decays slowly as $n\to\infty$, and $\sum_{n=1}^{\infty}\rho(n)=\infty$. The increments of fBm $B_k^H-B_{k-1}^H$ and $B_{k+n}^H-B_{k+n-1}^H$ are long-range dependent for $H>\frac{1}{2}$ since for $\alpha=2H-2$, as $n\to\infty$

$$\frac{1}{2}\left((n+1)^{2H}+(n-1)^{2H}-2n^{2H}\right)\approx H\left(2H-1\right)n^{2H-2}$$

Integral representations

Moving average representation

One of the earliest representations of fBm is due to Mandelbrot (see [MN68]), who uses a infinite moving average. In this particular representation, the process $Z(t)$ is a fBm

$$\begin{aligned} Z(t) &= \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}\left((t-s)_+^{H-\frac{1}{2}}-(-s)_+^{H-\frac{1}{2}}\right)dB_s \\ &= \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{-\infty}^{0}\left((t-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}\right)dB_s+\int_0^t (t-s)^{H-\frac{1}{2}}\,dB_s\right)\end{aligned}$$

where $(\cdot)_+\equiv\max(\cdot,0)$. A key issue with with using this representation for pricing is that you need to know the entire past of the process.

Integral representations on a finite interval for $H>\frac{1}{2}$

Some integral representations of fBm on finite intervals are only valid for a subset of the Hurst parameters. If $H\in(0.5,1)$ then the fBm is smoother (and more predictable) than standard Brownian motion and there are several finite-interval representations for this case (see [DU99, Dec03]). There are a couple of main steps in the derivation:

1. First derive an integral representation for the fBm covariance function $R_H(t,s)$ (equation (7)).
2. Next show a way to express the integrand of the representation of $R_H(t,s)$ as the product of identical functions $K_H(t,s)$ (though possibly with different parameters). This step involves lots of ugly manipulations.

3. With $K_H(t,s)$ in hand, define an inner product for functions, by multiplying each function by a version of $K_H(t,s)$ and integrating.
4. Finally, using the Itô isometry, the inner product of fBm processes (in L^2_Ω - i.e. their covariance) can be related to the inner product on L^2, in other words, an isometry is derived. In the simplest case, the functions used are step functions, e.g. $B_H(t) = \int_0^T \mathbb{1}_{[0,t]} dB_H$.

The key function from the second step (sometimes referred to as the reproducing kernel) is

$$K_H(t,s) = c_H \left(s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right) \tag{9}$$

(where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{-\frac{1}{2}}$ and $t > s$). It satisfies (by changing the order of integration[9])

$$\begin{aligned}
\int_0^{t\wedge s} K_H(t,u) K_H(s,u) du &= c_H^2 \int_0^{t\wedge s} \left(u^{\frac{1}{2}-H} \int_u^t (y-u)^{H-\frac{3}{2}} y^{H-\frac{1}{2}} dy \right) \left(u^{\frac{1}{2}-H} \int_u^s (x-u)^{H-\frac{3}{2}} x^{H-\frac{1}{2}} dx \right) du \\
&= c_H^2 \int_0^t \int_0^s (yx)^{H-\frac{1}{2}} \left(\int_0^{y\wedge x} u^{1-2H} (y-u)^{H-\frac{3}{2}} (x-u)^{H-\frac{3}{2}} du \right) dy dx \\
&= c_H^2 \beta(2-2H, H-\frac{1}{2}) \int_0^t \int_0^s |y-x|^{2H-2} dy dx \\
&= R_H(t,s)
\end{aligned}$$

The third step uses $K_H(t,s)$ to construct an operator on functions, denoted by K_H^*. Then for a function φ the representation for general $B_H(\varphi)$ is given in terms of standard Brownian motion as

$$B(\varphi) = \int_0^T (K_H^* \varphi)(v) dB_{\frac{1}{2}}(v) \tag{10}$$

and in particular, for plain vanilla fBm on over an interval

$$\begin{aligned}
B(\mathbb{1}_{[0,t]}(u)) &= \int_0^T \left(K_H^* \mathbb{1}_{[0,t]}(u) \right)(v) dB_{\frac{1}{2}}(v) \\
&= \int_0^t K_H(t,v) dB_{\frac{1}{2}}(v)
\end{aligned}$$

Reproducing fBm as a random walk

This last result suggests a way to generate fBm. Given an increasing sequence π_n of partitions of $[0,T]$, where the mesh size goes to zero as $n \to \infty$, the sequence of processes W^n

$$W^n \equiv \sum_{i=1}^n \frac{B_{t_{i+1}} - B_{t_i}}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} K_H(t,s) ds \tag{11}$$

converges to $B_H(t)$, and this can be used to simulate fBm (see [Dec03]). Examples are shown in Figure 5, using 80 steps per unit interval.

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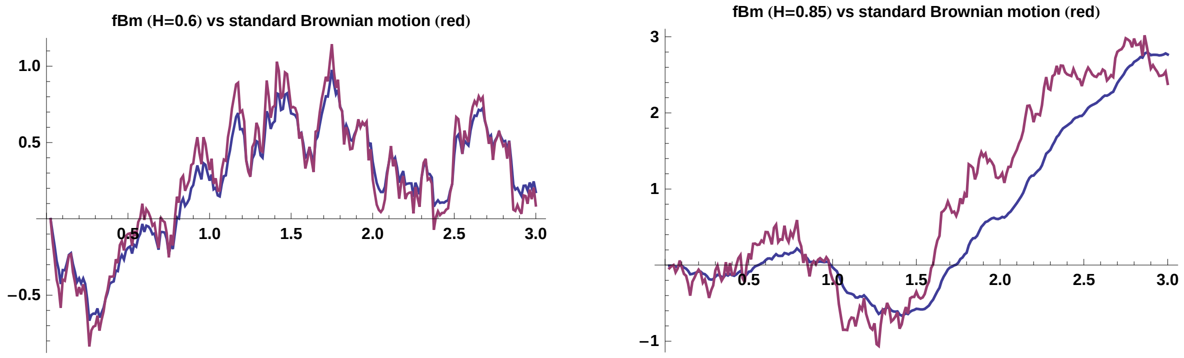
[9] And using the fact that $|r-u|^{2H-2} = \frac{(ru)^{H-\frac{1}{2}}}{\beta(2-2H, H-\frac{1}{2})} \int_0^{r\wedge u} v^{1-2H} (r-v)^{H-\frac{3}{2}} (u-v)^{H-\frac{3}{2}} dv$

Figure 5: Simulations of Fractional Brownian Motion

The graphs below show two simulations of fractional Brownian motion (in blue) for different values of the Hurst parameter, along with the corresponding standard Brownian motion (in red) used to construct the simulation. Each simulation is based on 80 steps per unit interval over $[0, 3]$, and uses the reproducing Kernel method detailed in equation (11).



fBm (H=0.6) vs standard Brownian motion (red)

fBm (H=0.85) vs standard Brownian motion (red)

Hurst Parameter Estimates

The OFHEO[10] unadjusted monthly housing index values from January 1991 - March 2008 were used to estimate the Hurst parameter for housing prices. Statistical software used to make the estimates is based on the R language [Tea05] extended with a number of packages designed to analyze long-memory time series. The Hurst parameter estimates were made using an implementation of the rescaled range ("R/S") method (see [Die02, Cle06, TTW95]). Time series methods were also used for comparison.

The OFHEO housing price index is constructed via the repeat-sales method, first proposed by Bailey, Muth, and Nourse [BMN63], using a modified version of the Case-Shiller [CS89] methodology. Data are gathered on repeat sales or refinancings of single-family properties whose mortgages have been purchased or securitized by Fannie Mae or Freddie Mac since January 1975. The index is also weighted, resulting in a measure of average price changes on the same properties.

The Hurst parameter estimated from the aggregate US housing prices is $H = 0.6263$ with a standard error of 0.0302, and the results of the R/S analysis are shown in Figure 6.

The available OFHEO data represent only 206 data points, so the estimates based on the curve fitting R/S technique should be considered rough. For comparison, the same data were subjected to a time-series analysis, under the assumption that the returns at each period could be represented by $r_t = \mu + \xi_t + \psi_1 \xi_{t-1} + \psi_2 \xi_{t-2} + \ldots$ where the innovations ξ_i are identical Gaussian variables (not independent) and $\sum_i \psi_i^2 < \infty$ (see [KYM07], and note the similarity of this representation with equation(11)). The fBm assumption then specifies the auto-covariance structure (see equation (8)) and an MLE can be made for the equivalent Hurst parameter. The result of this analysis for the US aggregate data yields an estimated Hurst parameter $H = 0.891$ with a standard error of 0.048. In this case the R^2 of the fit is 56.16%. A q-q plot of the residuals suggests that they are Gaussian.

For completeness, regional Hurst parameter estimates and their standard errors were calculated from the unadjusted OFHEO data using the R/S method, and the respective fits to the data are shown in Table (1) below.

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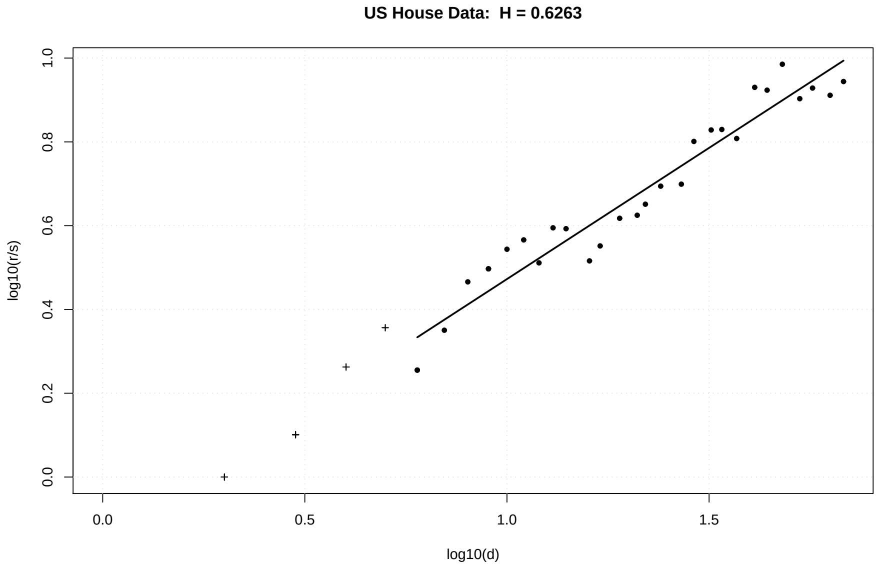
[10] See `http://www.ofheo.gov/hpi_download.aspx`

Figure 6: Hurst Parameter Estimates for US Aggregate House Price Index

The monthly OFHEO house price indices (unadjusted) were used to estimate the Hurst parameter for aggregate house prices in in the United States., The index data was transformed to per-period returns prior to analysis, consistent with the assumption that the house prices can be modeled by a geometric fBm. The chart below shows the log-log plot of the R/S statistic (or "rescaled adjusted range") versus the lag d, where $\mathbb{E}\left[\frac{R}{S}(d)\right] \sim C_H d^H$ as $n \to \infty$ for some constant C_H (see [TTW95]). There is a transient zone at the low end of the plot which is not used for fitting, and these points are indicated with a + symbol.



Pricing Contingent Claims Using fBm

By extension from equation (1), assume that the underlying price process can be described by a simple geometric fractional random-walk

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H \tag{12}$$

with Hurst parameter H, return μ, volatility σ, and where the risk-free return is r.

The stochastic integral of equation (12) can be defined path-wise, as a refinement of Riemann-Stieltjes integrals, since, for $H \in (0.5, 1)$ the paths are smoother than standard Brownian motion. While fBm is not a martingale, there is a measure Q, equivalent to the real-world measure under which the current price is equal to the expected value of the discounted future price, conditional on the current price. By contrast with the theory for standard Brownian motion, the conditional expectations change with time. Hedging portfolios may then be non-anticipative functionals of the price paths, though they are not Markov.

In spite of the issues that fBm raises, pricing European contingent claims under fBm is quite similar to the Black-Scholes-Merton model. Assuming the claim expires at time T, paying payoff $(C(S_T, K))$, the value of the claim at any time $t = 0$ is

$$V(S_0, 0, K, T) = \mathbb{E}^{Q}_{0,\mathrm{S}_0}\left[e^{-rT} C(S_T, K)\right]$$

	Estimate	Std.Err
East North Central	0.5391	0.0358
East South Central	0.5405	0.0361
Middle Atlantic	0.4434	0.0376
Mountain	0.9679	0.0994
New England	0.4565	0.0619
Pacific	0.3877	0.0628
South Atlantic	0.4639	0.0548
West North Central	0.7630	0.0727
West South Central	0.5006	0.0282
USA	0.6263	0.0302

Table 1: Regional Hurst Parameter Estimates (R/S method)

and as fBm is a centered Gaussian process, the value can be written as

$$V(S_0, 0, K, T) = e^{-rT} \int_{-\infty}^{\infty} C\left(S_0 e^{rT - \frac{\left(\sigma T^H\right)^2}{2} + \sigma T^H y}, K\right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$

which for a vanilla call option is

$$V(S_0, 0, K, T) = S_0 N(d_1) - K e^{-rT} N(d_2)$$

with

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_0}{K} + rT + \frac{\left(\sigma T^H\right)^2}{2}}{\sigma T^H} \\ d_2 &= \frac{\ln \frac{S_0}{K} + rT - \frac{\left(\sigma T^H\right)^2}{2}}{\sigma T^H} \end{aligned}$$

which converges to the Black-Scholes-Merton formula for a vanilla call as $H \searrow 0.5$. Further details can be found in [Sot25, SV01, SV03][Sot25, SV01, SV03].